

Type-II Matrices and Combinatorial Structures

Ada Chan and Chris Godsil

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Abstract

Type-II matrices are a class of matrices used by Jones in his work on spin models. In this paper we show that type-II matrices arise naturally in connection with some interesting combinatorial and geometric structures.

1 Introduction

If M and N are matrices of the same order, their *Schur product* is the matrix $M \circ N$, defined by the condition

$$(M \circ N)_{i,j} = M_{i,j}N_{i,j}.$$

The Schur product is commutative and associative, with an identity element J , the all-ones matrix. If $M \circ N = J$ we say that N is the *Schur inverse* of M , and denote it $M^{(-)}$.

A *type-II matrix* is a Schur invertible $n \times n$ matrix W over \mathbb{C} such that

$$WW^{(-)T} = nI.$$

This condition implies that W^{-1} exists and

$$W^{(-)T} = nW^{-1}.$$

In [6] Jones showed that certain special type-II matrices could be used to construct so-called spin models, which could in turn be used to construct

interesting invariants of knots and links (including the Jones polynomial). The main goal of this paper is to show that type-II matrices are much more common than might be expected: in particular they arise in connection with a range of combinatorial and geometric structures: symmetric designs, sets of equiangular lines and strongly regular graphs.

2 The Basics

We offer some examples of type-II matrices. First

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is a symmetric type-II matrix. If ω is a primitive cube root of unity then

$$\begin{pmatrix} 1 & 1 & \omega \\ \omega & 1 & 1 \\ 1 & \omega & 1 \end{pmatrix}$$

is also type-II. For any non-zero complex number t , the matrix

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & t & -t \\ 1 & -1 & -t & t \end{pmatrix}$$

is type-II. Next we have the *Potts models*: if W is $n \times n$ and

$$W = (t - 1)I + J,$$

then

$$\begin{aligned} WW^{(-)T} &= ((t - 1)I + J)((t^{-1} - 1)I + J) \\ &= ((2 - t - t^{-1})I + (n - 2 + t + t^{-1})J), \end{aligned}$$

whence it follows that W is type-II whenever $2 - t - t^{-1} = n$, i.e., whenever t is a root of the quadratic

$$t^2 + (n - 2)t + 1.$$

As the first example suggests, any Hadamard matrix is a type-II matrix, and it is not unreasonable to view type-II matrices as a generalisation of Hadamard matrices.

The Kronecker product of two type-II matrices is a type-II matrix; this provides another easy way to increase the supply of examples. Recall that a *monomial matrix* is the product of a permutation matrix and a diagonal matrix. It is straightforward to verify that if W is type-II and M and N are invertible monomial matrices, then MWN is type-II. We say W' is *equivalent* to W if $W' = MWN$, where M and N are invertible monomial matrices.

The transpose W^T is also type-II, as is $W^{(-)}$, but these matrices may not be equivalent to W . It would be a useful exercise to prove that any 2×2 type-II matrix is equivalent to the first example above, any 3×3 type-II matrix is equivalent to the second, and any 4×4 type-II matrix is equivalent to a matrix in the third family.

Let W be a Schur-invertible matrix, with rows and columns indexed by the set Ω , where $|\Omega| = n$. Let the vectors

$$e_a, \quad a \in \Omega$$

denote the standard basis for \mathbb{C}^Ω . We define a set of n^2 vectors in \mathbb{C}^n as follows.

$$Y_{a,b} := W e_a \circ W^{(-)} e_b.$$

We can view $Y_{a,b}$ as the Schur ratio of the a - and b -columns of W . The *Nomura algebra* \mathcal{N}_W of W consists of the set of $n \times n$ complex matrices M such that each of the n^2 vectors $Y_{a,b}$ is an eigenvector for M . The Nomura algebra is non-empty, because it always contains I .

2.1 Lemma. *Let W be a Schur invertible and invertible matrix. Then W is a type-II matrix if and only if $J \in \mathcal{N}_W$.*

Proof. Let D_a be the $n \times n$ diagonal matrix such that

$$(D_a)_{i,i} := W_{i,a}.$$

Since W is invertible, its columns

$$W e_b, \quad b \in \Omega$$

are linearly independent. Since D_a is invertible and

$$Y_{a,b} = D_a^{-1} W e_b,$$

we see that the vectors

$$Y_{a,b}, \quad b \in \Omega$$

are linearly independent and consequently they form a basis for \mathbb{C}^n .

Now $Y_{a,a} = \mathbf{1}$, so $J \in \mathcal{N}_W$ if and only if

$$JY_{a,b} = n\delta_{ab}Y_{a,b},$$

equivalently

$$\sum_r \frac{W_{r,a}}{W_{r,b}} = (W^{(-)T}W)_{b,a} = n\delta_{b,a}$$

for all $a, b \in \Omega$. □

It follows that if W is a type-II matrix of order $n \times n$, then \mathcal{N}_W contains I and J and $\dim \mathcal{N}_W \geq 2$ when $n \geq 2$. We say that \mathcal{N}_W is *trivial* if $\dim \mathcal{N}_W = 2$. All the work in this paper is motivated by the desire to find type-II matrices with non-trivial Nomura algebras. One reason is that if W is a type-II matrix and $W \in \mathcal{N}_W$, then we may use W to construct a link invariant. The Potts model, which we mentioned above, has this property and the corresponding link invariants are evaluations of the Jones polynomial. For more on this connection, see [5] and [4].

A type-II matrix W such that $W \in \mathcal{N}_W$ is known as a *spin model*. The Potts model aside, very few interesting spin models are known. If W is a spin model other than the Potts model, then \mathcal{N}_W contains I , J and W , and therefore $\dim \mathcal{N}_W \geq 3$. Spin models have proved very difficult to find. Hence we are lead to search for type-II matrices whose Nomura algebras are non-trivial. For reasons that are not at all clear, even these seem to be scarce.

The previous discussion glosses over one point. If W_1 and W_2 are type-II matrices, then the Nomura algebra of $W_1 \otimes W_2$ is the tensor product of the Nomura algebras of W_1 and W_2 . Since

$$\dim(\mathcal{N}_{W_1 \otimes W_2}) = \dim(\mathcal{N}_{W_1}) \dim(\mathcal{N}_{W_2}),$$

the Nomura algebra of $W_1 \otimes W_2$ is always non-trivial. However the corresponding link invariants are of no interest, since they are built in an obvious way from the invariants belonging to the factors. Therefore our search is actually for type-II matrices which have non-trivial Nomura algebra and which are not equivalent to Kronecker products of type-II matrices.

3 Nomura Algebras

We have introduced type-II matrices and their Nomura algebras. Now we describe the connection between type-II matrices and combinatorics; the connection is mediated by association schemes.

Let W be a type-II matrix of order $n \times n$. We saw in the previous section that

$$Y_{a,b}, \quad b \in \Omega$$

form a basis for \mathbb{C}^n . If $M \in \mathcal{N}_W$, then the matrix representing M relative to this basis is diagonal, from which we conclude that if $M, N \in \mathcal{N}_W$ then $MN = NM$. In other words, the Nomura algebra of a type-II matrix is commutative. We will also see that it is closed under the Schur product.

Let W be a type-II matrix, with rows and columns indexed by the set Ω , where $|\Omega| = n$. If $M \in \mathcal{N}_W$, there is an $n \times n$ matrix $\Theta_W(M)$ such that

$$MY_{a,b} = (\Theta_W(M))_{a,b} Y_{a,b}.$$

We call $\Theta_W(M)$ the *matrix of eigenvalues* of M . (When no confusion will result, we write $\Theta(M)$ rather than $\Theta_W(M)$.) Note that

$$\Theta(MN) = \Theta(M) \circ \Theta(N).$$

Also Θ is an injective linear map from \mathcal{N}_W into the space of $n \times n$ complex matrices.

We define a second family of n^2 of vectors in \mathbb{C}^n as follows.

$$Y'_{a,b} := W^T e_a \circ W^{(-)T} e_b$$

Thus $Y'_{a,b}$ is the Schur ratio of two columns of W^T , and so the set of matrices with the vectors $Y'_{a,b}$ as eigenvectors is \mathcal{N}_{W^T} . The following critical result is due to Nomura [7]; it shows that the image of \mathcal{N}_W under Θ is contained in \mathcal{N}_{W^T} .

3.1 Theorem. *If $M \in \mathcal{N}_W$ then*

$$\Theta(M)Y'_{s,r} = nM_{r,s}Y'_{s,r}.$$

Proof. Suppose

$$F_i := \frac{1}{n} Y_{u,i} Y_{i,u}^T.$$

We verify easily that

$$F_i F_j = \delta_{i,j} F_i,$$

which shows that the F_i 's form an orthogonal set of n idempotents. We note that $\text{rk}(F_i) = 1$ and $\text{tr}(F_i) = 1$. As the F_i 's commute it follows that $\sum_i F_i$ is an idempotent matrix with trace equal to n ; hence

$$\sum_i F_i = I.$$

We have

$$M F_i = \frac{1}{n} M Y_{u,i} Y_{i,u}^T = (\Theta(M))_{u,i} F_i.$$

Summing this over i in Ω , recalling that $\sum_i F_i = I$, we get

$$M = \sum_i (\Theta(M))_{u,i} F_i. \quad (1)$$

Now

$$(F_i)_{r,s} = \frac{1}{n} \frac{W_{r,u}}{W_{r,i}} \frac{W_{s,i}}{W_{s,u}} = \frac{1}{n} \frac{W_{r,u}}{W_{s,u}} \frac{W_{s,i}}{W_{r,i}}$$

and therefore, by (1),

$$M_{r,s} = \frac{1}{n} \frac{W_{r,u}}{W_{s,u}} \sum_i (\Theta(M))_{u,i} \frac{W_{s,i}}{W_{r,i}}.$$

Hence

$$n M_{r,s} (Y'_{s,r})_u = (\Theta(M) Y'_{s,r})_u,$$

which implies the theorem. \square

It is an easy consequence that the \mathcal{N}_W is closed under the Schur product.

We describe a simple way to test if two eigenvectors $Y_{a,b}$'s belong to the same eigenspace of \mathcal{N}_W .

3.2 Lemma. *If $Y_{a,u}^T Y_{b,c} \neq 0$ then $(\Theta(M))_{u,a} = (\Theta(M))_{b,c}$.*

Proof. It follows from $\sum_i F_i = I$ that

$$Y_{b,c} = \frac{1}{n} \sum_i (Y_{i,u}^T Y_{b,c}) Y_{u,i}.$$

So

$$(\Theta(M))_{b,c} Y_{b,c} = M Y_{b,c} = \frac{1}{n} \sum_i (Y_{i,u}^T Y_{b,c}) (\Theta(M))_{u,i} Y_{u,i}.$$

Multiply both sides of this by $Y_{a,u}^T$ to get

$$\begin{aligned} (\Theta(M))_{b,c} Y_{a,u}^T Y_{b,c} &= \frac{1}{n} (Y_{a,u}^T Y_{b,c}) (\Theta(M))_{u,a} Y_{a,u}^T Y_{u,a} \\ &= Y_{a,u}^T Y_{b,c} (\Theta(M))_{u,a}. \end{aligned}$$

If $Y_{a,u}^T Y_{b,c} \neq 0$, this implies that $(\Theta(M))_{u,a} = (\Theta(M))_{b,c}$. \square

4 Association Schemes

We recall some definitions. An *association scheme* with d classes is a collection \mathcal{A} of 01-matrices A_0, \dots, A_d of order $n \times n$ such that:

- (a) $A_0 = I$.
- (b) $\sum_i A_i = J$.
- (c) $A_i^T \in \mathcal{A}$ for $i = 0, \dots, d$.
- (d) The product $A_i A_j$ lies in the span of \mathcal{A} , for all i and j .
- (e) $A_i A_j = A_j A_i$.

The matrices A_i are the adjacency matrices of directed graphs whose arc sets partition the arcs of the complete directed graph on n vertices. It follows from the axioms that $A_i J = J A_i$, whence each directed graph is regular. The span of \mathcal{A} is called the *Bose-Mesner algebra* of the association scheme. Since the A_i are 01-matrices and sum to J , they form a basis for \mathcal{A} ; since the set consisting of \mathcal{A} and the zero matrix is closed under Schur product, it follows that the span of \mathcal{A} is closed under the Schur product. The axioms also insure that the Bose-Mesner algebra is closed under transpose and under matrix multiplication. On the other hand, a vector space of matrices is the Bose-Mesner algebra of an association scheme if it contains I and J and is closed under transpose, matrix and Schur product, and it is commutative with respect to matrix multiplication. See [1] for details.

The simplest example of an association scheme arises if we take $d = 1$ and $A_1 = J - I$. This is the association scheme of the complete graph (and the Nomura algebra of a Potts model).

4.1 Corollary. *If W is a type-II matrix, then \mathcal{N}_W is the Bose-Mesner algebra of an association scheme.*

Proof. It is immediate from its definition that \mathcal{N}_W is closed under and is commutative with respect to matrix multiplication. By Lemma 2.1, \mathcal{N}_W contains I and J . We show that it is also closed under transpose and Schur multiplication.

Theorem 3.1 yields that if $M \in \mathcal{N}_W$ then

$$\Theta_{W^T}(\Theta_W(M)) = nM^T.$$

Since $\Theta_W(M) \in \mathcal{N}_{W^T}$ we see that $\Theta_{W^T}(\Theta_W(M)) \in \mathcal{N}_W$. Therefore \mathcal{N}_W is closed under transpose. We also see that Θ_{W^T} is surjective.

We saw that if $M, N \in \mathcal{N}_{W^T}$ then

$$\Theta_{W^T}(MN) = \Theta_{W^T}(M) \circ \Theta_{W^T}(N).$$

This shows that $\Theta_{W^T}(\mathcal{N}_{W^T})$ is closed under Schur multiplication. Since Θ_{W^T} is surjective, it follows that \mathcal{N}_W is closed under Schur multiplication. \square

If W is a type-II matrix with algebra \mathcal{N}_W then, as noted before, W determines a spin model if and only if W lies in \mathcal{N}_W . As any type-II matrix equivalent to W has the isomorphic Nomura algebra, [5], we may concentrate on the matrices W that lie in their Nomura algebra. If $W \in \mathcal{N}_W$ then

- (a) W is normal.
- (b) The diagonal of W is constant, that is, $W \circ I = cI$ for some c .
- (c) The row and column sums of W are all equal.

These conditions hold because they are satisfied by any matrix in a Bose-Mesner algebra.

One consequence of Nomura's theorem is that, when searching for spin models, we can restrict ourselves to type-II matrices that lie in the Bose-Mesner algebra of an association scheme. This is important because there may be uncountably many type-II matrices of a given order n , but there are only finitely many association schemes of order n . Hence our search space is considerably restricted.

5 Hadamard Matrices

A Hadamard matrix is a ± 1 -matrix of order $n \times n$ such that

$$H^T H = nI.$$

Since $H \circ H = J$ it follows that H is a type-II matrix. Hadamard matrices have long been of interest to combinatorialists. Since they are the simplest examples of type-II matrices, we summarise what is known about their Nomura algebras here.

5.1 Lemma. *If W is real then all matrices in \mathcal{N}_W are symmetric.*

Proof. If W is real then the eigenvectors $Y_{a,b}$ are real. Hence the Schur idempotents of the scheme have only real eigenvalues. Since \mathcal{N}_W is closed under transposes and is a commutative algebra, the Schur idempotents are real normal matrices. A real normal matrix is symmetric if and only if its eigenvalues are real. \square

The following is a new proof of a result due to Jaeger et al [5].

5.2 Lemma. *Let W be a Hadamard matrix of order n . If \mathcal{N}_W is non-trivial, then n is divisible by eight.*

Proof. Let w_i denote We_i . Normalise W so that $w_1 = \mathbf{1}$ and assume $1, i, j$ and k are distinct. Then

$$(w_1 + w_i) \circ (w_1 + w_j) \circ (w_1 + w_k)$$

is the Schur product of three vectors with entries $0, \pm 2$. The sum of the entries of this vector is

$$\begin{aligned} \langle \mathbf{1}, w_1^{\circ 3} \rangle + \langle \mathbf{1}, w_1^{\circ 2} \circ (w_i + w_j + w_k) \rangle \\ + \langle \mathbf{1}, w_1 \circ (w_i \circ w_j + w_i \circ w_k + w_j \circ w_k) \rangle + \langle \mathbf{1}, w_i \circ w_j \circ w_k \rangle \end{aligned} \quad (2)$$

Since W is a Hadamard matrix, the second and third terms here are zero, whence we deduce that, modulo 8,

$$n + \langle \mathbf{1}, w_i \circ w_j \circ w_k \rangle = 0$$

and therefore, if n is not divisible by 8, then $Y_{i,1} = w_i$ cannot be orthogonal to $Y_{j,k} = w_j \circ w_k$. \square

If H is a Hadamard matrix of order less than 32, its Nomura algebra is a product of Potts models. (Unpublished computations by Allan Roberts and the second author.)

Hadamard matrices form a special class of a more general class of type-II matrices. A complex matrix is *flat* if all its entries have the same absolute value. The following result is easy to prove.

5.3 Lemma. *For an $n \times n$ matrix, any two of the following statements imply the third:*

(a) W is a type-II matrix.

(b) $n^{-1/2}W$ is unitary.

(c) $|W_{i,j}| = 1$ for all i and j . □

In other words, a unitary matrix is type-II if and only if it is flat. The character table of an abelian group is flat, type-II and unitary. Flat unitary matrices appear in quantum physics in connection to mutually unbiased sets of orthogonal bases.

6 Symmetric Designs

We consider type-II matrices with exactly two distinct entries, that are not Hadamard matrices.

6.1 Theorem. *Suppose $W = aJ + (b - a)N$, where N is a 01-matrix and $a \neq \pm b$. Then W is type II if and only if N is the incidence matrix of a symmetric design.*

Proof. Let N be the incidence matrix of a symmetric (v, k, λ) -design, and let W be given by

$$W = J + (t - 1)N,$$

where

$$t = \frac{1}{2(k - \lambda)} \left(2(k - \lambda) - v \pm \sqrt{v(v - 4(k - \lambda))} \right).$$

We show that W is a type-II matrix.

We have

$$W^{(-)} = (t^{-1} - 1)N + J$$

and, as $NJ = N^T J = kJ$ and $J^2 = vJ$,

$$\begin{aligned} WW^{(-)T} &= (t-1)(t^{-1}-1)NN^T + (k(t+t^{-1}-2)+v)J \\ &= (t-1)(t^{-1}-1)(k-\lambda)I + ((k-\lambda)(t+t^{-1}-2)+v)J. \end{aligned}$$

The coefficient of J is zero if

$$(k-\lambda)(t-1)^2 + v(t-1) + v = 0,$$

which yields sufficiency.

We now prove the converse. If W has exactly two distinct entries, there is no harm in assuming that we have

$$W = J + (t-1)N$$

for some 01-matrix N and some complex number t such that $t \neq \pm 1$. Then $W^{(-)T} = J + (t^{-1}-1)N^T$ and so, if W is $v \times v$, we have

$$WW^{(-)T} = vJ + (t-1)NJ + (t^{-1}-1)JN^T + (t-1)(t^{-1}-1)NN^T.$$

Since $WW^{(-)T} = vI$ and NN^T is symmetric, this implies that

$$M := (t-1)NJ + (t^{-1}-1)JN^T$$

is symmetric. We work with this. Note that this equation yields

$$M - M^T = (t-t^{-1})NJ + (t^{-1}-t)JN^T = (t-t^{-1})(NJ - (NJ)^T).$$

Since $M = M^T$ and $t \neq \pm 1$, this forces us to conclude that NJ is symmetric. Hence there is a positive integer k such that

$$NJ = JN^T = kJ.$$

Returning to our expression for $WW^{(-)T}$, we now have

$$WW^{(-)T} = (v + k(t+t^{-1}-2))J + (2-t-t^{-1})NN^T. \quad (3)$$

Since $(2-t-t^{-1}) = -(t-1)^2/t$ and $t \neq 1$, it follows that NN^T is a linear combination of I and J , and consequently N is the incidence matrix of a symmetric design. \square

Note that if $v + k(t + t^{-1} - 2) = 0$ in 3) then we get $NN^T = kI$. Since N is a square 01-matrix, $NN^T = kI$ only when $k = 1$. In this case, N is the incidence matrix of the complement of the complete design, and $W = J + (t - 1)N$ is equivalent to the Potts model.

If H is a Hadamard matrix, we may multiply it fore and aft by diagonal matrices, thus setting all entries in the first row and column to 1. If H_1 is the matrix we get from this by deleting the first row and column, then

$$\frac{1}{2}(H_1 + J)$$

is the incidence matrix of a symmetric design. This gives a large class of examples of symmetric designs.

6.2 Lemma. *Suppose W is a type-II matrix of the form $(t - 1)N + J$, where N is the incidence matrix of a symmetric (v, k, λ) -design. If $v > 3$, then all matrices in \mathcal{N}_{W^T} are symmetric.*

Proof. We show that $\langle Y_{i,j}, Y_{i,j} \rangle \neq 0$ when $v > 3$. By Lemma 3.2, it follows that $\Theta(M)_{i,j} = \Theta(M)_{j,i}$ for all M in \mathcal{N}_W and for all i and j .

We have

$$\begin{aligned} \langle Y_{i,j}, Y_{i,j} \rangle &= (k - \lambda)(t^2 + t^{-2}) + v - 2k + 2\lambda \\ &= (k - \lambda)(t^2 - 2 + t^{-2}) + v, \end{aligned}$$

and so, if $\langle Y_{i,j}, Y_{i,j} \rangle = 0$ then

$$t^2 - 2 + t^{-2} = \frac{-v}{(k - \lambda)}.$$

From our computations in the proof of the previous theorem,

$$(k - \lambda)(t - 2 + t^{-1}) + v = 0, \tag{4}$$

and so

$$t - 2 + t^{-1} = \frac{-v}{(k - \lambda)}.$$

As

$$t^2 - 2 + t^{-2} = (t - 2 + t^{-1})(t + 2 + t^{-1}),$$

these equations imply that, if $\langle Y_{i,j}, Y_{i,j} \rangle = 0$, then

$$t + 1 + t^{-1} = 0,$$

whence (4) implies that $v = 3(k - \lambda)$.

Since $v(v - 1)\lambda = vk(k - 1)$, if $v = 3(k - \lambda)$, then

$$k^2 = k + (v - 1)\lambda = (3\lambda + 1)(k - \lambda)$$

and therefore

$$k^2 - (3\lambda + 1)k + 3\lambda^2 + \lambda = 0$$

This discriminant of this quadratic is

$$1 + 2\lambda - 3\lambda^2 = (1 - \lambda)(1 + 3\lambda),$$

which is negative if $\lambda > 1$. The lemma follows. \square

6.3 Lemma. *Let N be the incidence matrix of a symmetric design, and let W be a type-II matrix of the form $(t - 1)N + J$. If $t \neq -1$, then the difference of two distinct columns of N is an eigenvector for the Nomura algebra of W .*

Proof. If u is a point in the design and α and β are the i -th and j -th blocks in the design, then

$$(Y_{i,j})_u = \begin{cases} t, & \text{if } u \in \alpha \setminus \beta; \\ t^{-1}, & \text{if } u \in \beta \setminus \alpha; \\ 1, & \text{otherwise.} \end{cases}$$

By the previous lemma, $Y_{i,j}$ and $Y_{j,i}$ have the same eigenvalues for any matrix in \mathcal{N}_W . Therefore the vector

$$(t - t^{-1})^{-1}(Y_{i,j} - Y_{j,i})$$

is an eigenvector for each matrix in \mathcal{N}_W , but this vector is just the difference of the i -th and j -th columns of N . \square

We note that if $t = -1$ then $(t - 1)N + J$ is type II if and only if it is a Hadamard matrix. The previous lemmas lead to the following disappointing consequence.

6.4 Theorem. *Suppose W is a type-II matrix of the form $(t - 1)N + J$, where N is the incidence matrix of a symmetric (v, k, λ) -design. If $v > 3$ and $t \neq -1$, then the Nomura algebra of W is trivial.*

Let $Z_{i,j} := Ne_i - Ne_j$ for some $i \neq j$. If k is distinct from i and j then

$$\langle Z_{i,j}, Ne_k \rangle = \langle Ne_i, Ne_k \rangle - \langle Ne_j, Ne_k \rangle = \lambda - \lambda = 0$$

while

$$\langle Z_{i,j}, Ne_i \rangle = k - \lambda.$$

We conclude that $\langle Z_{i,j}, Z_{i,k} \rangle = k - \lambda$ and therefore at least one of

$$Y_{i,k}^T Y_{i,j}, \quad Y_{k,i}^T Y_{i,j}, \quad Y_{i,k}^T Y_{j,i} \quad \text{and} \quad Y_{k,i}^T Y_{j,i}$$

is non-zero. It follows from Lemma 3.2 and Lemma 6.2 that

$$\Theta(M)_{i,k} = \Theta(M)_{i,j}$$

for any matrix M from \mathcal{N}_W . It follows that \mathcal{N}_W must be trivial. \square

7 Equiangular Lines

We consider sets of lines in \mathbb{C}^d . A set of lines in \mathbb{C}^d spanned by the unit vectors x_1, \dots, x_n is *equiangular* if there is a real number α such that

$$|\langle x_i, x_j \rangle| = \alpha$$

whenever $i \neq j$. Note that it is reasonable to take the absolute value here, because if $\lambda \in \mathbb{C}$ and $|\lambda| = 1$ then λx_i and x_i are unit vectors spanning the same line. We will refer to α as the *angle* between the lines. We are also interested in equiangular sets of lines in \mathbb{R}^d ; the above definition still works in this case. We have the following result, due to [8].

7.1 Theorem. *If there is a set of n equiangular lines in \mathbb{C}^d or \mathbb{R}^d with angle α and $d\alpha^2 < 1$, then*

$$n \leq \frac{d(1 - \alpha^2)}{1 - d\alpha^2}.$$

Proof. Suppose x_1, \dots, x_n are unit vectors spanning a set of equiangular lines in \mathbb{C}^d and suppose $X_i := x_i x_i^*$. Then X_i is a Hermitian matrix that represents orthogonal projection onto the line spanned by x_i . Assume that $|\langle x_i, x_j \rangle| = \alpha$ when $i \neq j$. The space of Hermitian matrices is a real inner product space with inner product $\langle X, Y \rangle$ given by

$$\langle X, Y \rangle = \text{tr}(XY).$$

Then $\langle X_i, X_i \rangle = 1$ and if $i \neq j$ then

$$\begin{aligned}\langle X_i, X_j \rangle &= \text{tr}(X_i X_j) = \text{tr}(x_i x_i^* x_j x_j^*) \\ &= \text{tr}(x_j^* x_i x_i^* x_j) \\ &= |x_i^* x_j|^2 \\ &= \alpha^2.\end{aligned}$$

If

$$Z := \sum_i X_i$$

then

$$\langle Z, Z \rangle = n + (n^2 - n)\alpha^2$$

and if $\gamma \in \mathbb{R}$, then

$$\langle Z - \gamma I, Z - \gamma I \rangle = n + (n^2 - n)\alpha^2 - 2\gamma n + \gamma^2 d.$$

Here the right side is a quadratic in γ , and is non-negative for all real γ . Its minimum value occurs when $\gamma = n/d$, which implies that

$$-\frac{n^2}{d} + n(1 + \alpha^2(n - 1)) \geq 0.$$

The theorem follows from this. \square

Note that the above proof still works if we replace \mathbb{C} by \mathbb{R} and ‘Hermitian’ by ‘symmetric’.

We say a set of lines is *tight* if equality holds in the bound of the previous theorem. We say that an $n \times n$ matrix C is a *generalized conference matrix* if:

- (a) C is Hermitian
- (b) $C_{i,i} = 0$ for all i .
- (c) $|C_{i,j}| = 1$ if $i \neq j$.
- (d) The minimal polynomial of C is quadratic.

Note that a conference matrix is an $n \times n$ matrix with diagonal entries zero and off-diagonal entries ± 1 , such that $CC^T = (n - 1)I$. It is known that a conference matrix is equivalent to a symmetric or skew symmetric conference matrix. If C is symmetric then it is Hermitian and $C^2 - (n - 1)I = 0$. If C is skew symmetric, then iC is Hermitian and $(iC)^2 - (n - 1)I = 0$.

7.2 Corollary. Suppose x_1, \dots, x_n are unit vectors that span a set of equiangular lines in \mathbb{C}^d with angle α and Gram matrix G , and suppose $G = I + \alpha C$. Then the set of lines is tight if and only if C is a generalized conference matrix.

Proof. Suppose x_1, \dots, x_n span a set of equiangular lines in \mathbb{C}^d , let X_i be the orthogonal projection onto the line spanned by x_i and set $Z = \sum_i X_i$. If this set of lines is tight, then

$$\langle Z - \gamma I, Z - \gamma I \rangle = 0$$

and consequently

$$\sum_i X_i = \frac{n}{d} I.$$

Let U be the $n \times d$ matrix with i -th row equal to x_i^* . Then

$$U^* U = \sum_i X_i = \frac{n}{d} I.$$

Now $G := UU^*$ is the Gram matrix of the unit vectors x_1, \dots, x_n ; since UU^* and U^*U have the same non-zero eigenvalues with the same multiplicities it follows that the eigenvalues of G are 0 and n/d . Since our set of lines is equiangular, we may write

$$G = I + \alpha C.$$

Here C is Hermitian, its diagonal entries are zero, its off-diagonal entries all have absolute value 1, and its minimal polynomial is quadratic. Thus it is a generalized conference matrix.

For the converse, suppose that C is a non-zero Hermitian matrix with zero diagonal and

$$C^2 - \beta C - \gamma I = 0.$$

Then the diagonal entries of C^2 are positive, whence $\gamma \neq 0$ and C is invertible. If τ is the least eigenvalue of C , then

$$G := I - \frac{1}{\tau} C$$

is Hermitian and all its eigenvalues non-negative. Assume $\text{rk}(G) = d$. Since $\text{tr}(G) = n$ it follows that the eigenvalues of G are 0 and n/d . Hence there is an $n \times d$ matrix U such that

$$U^* U = \frac{n}{d} I, \quad UU^* = G.$$

Thus G is Gram matrix of the columns of U^* , and so these columns span a set of equiangular lines in \mathbb{C}^d . Since $U^*U = (n/d)I$, the set of lines is tight. \square

Conditions (a) and (c) in the definition of generalized conference matrix imply that $(C^2)_{i,i} = (n-1)I$, whence the minimal polynomial of C has the form $z^2 - \beta z - (n-1)$, for some β .

7.3 Theorem. *Suppose C is a generalized conference matrix of order $n \times n$ with minimal polynomial $z^2 - \beta z - (n-1)$. If $t + t^{-1} + \beta = 0$, then $tI + C$ is type II.*

Proof. If C is a generalized conference matrix, then

$$(tI + C)^{(-)T} = t^{-1}I + C$$

and therefore

$$\begin{aligned} (tI + C)(tI + C)^{(-)T} &= I + t^{-1}C + tC^{(-)T} + CC^{(-)T} \\ &= I + (t + t^{-1})C + C^2 \\ &= I + (t + t^{-1})C + \beta I + (n-1)I \\ &= nI + (t + t^{-1} + \beta)C. \end{aligned}$$

Hence $tI + C$ is type-II if

$$t + t^{-1} + \beta = 0. \quad \square$$

We derive a converse to this result, under weaker conditions.

7.4 Theorem. *Let W be a type-II matrix with all diagonal entries equal to c and with quadratic minimal polynomial. If $W - cI$ is Hermitian, it is a scalar multiple of a generalized conference matrix.*

Proof. Suppose that W is $n \times n$ and

$$W^2 - \beta W - \gamma I = 0.$$

Since W is invertible, $\gamma \neq 0$ and

$$W^{-1} = -\frac{1}{\gamma}(\beta I - W).$$

Hence

$$J = nW \circ W^{-T} = -\frac{n}{\gamma}(\beta W \circ I - W \circ W^T),$$

from which we find that

$$W \circ W^T = \beta W \circ I + \frac{\gamma}{n} J. \quad (5)$$

It follows that all off-diagonal entries of W have the same absolute value (namely $\sqrt{\gamma/n}$). \square

8 Strongly Regular Graphs

A graph X is strongly regular if it is not complete and there are integers k , a and c such that the number of common neighbours of an ordered pair of vertices (u, v) is k , a or c according as u and v are equal, adjacent or distinct and not adjacent. Trivial examples are provided by the graphs mK_n and their complements. The Petersen graph provides a less trivial example. A strongly regular graph X is *primitive* if both X and its complement are connected; an imprimitive strongly regular graph is isomorphic to mK_n or its complement. A strongly regular graph X gives rise to an association scheme with two classes, corresponding to X and its complement. Conversely each association scheme with two classes determines a complementary pair of strongly regular graphs.

8.1 Theorem. *Let X be a primitive strongly regular graph with v vertices, valency k , and eigenvalues k , θ and τ , where $\theta > \tau$. Let A_1 be the adjacency matrix of X and A_2 the adjacency matrix of its complement. Suppose*

$$W := I + xA_1 + yA_2.$$

Then W is a type-II matrix if and only if one of the following holds

- (a) $y = x = \frac{1}{2}(2 - v \pm \sqrt{v^2 - 4v})$.
- (b) $x = 1$ and $y = 1 + \frac{1}{2(k-\lambda)}(-v \pm \sqrt{v^2 - 4(\bar{k} - \lambda)v})$ and A_2 is the incidence matrix of a symmetric (v, \bar{k}, λ) -design where $\bar{k} = v - k - 1$.
- (c) $x = -1$ and $y = \frac{1}{2}(\lambda \pm \sqrt{\lambda^2 - 4})$ (where $\lambda = (1 + \theta\tau)^{-1}(2 - 2\theta\tau - v)$), and A_1 is the incidence matrix of a symmetric design.

(d) $x + x^{-1}$ is a zero of the quadratic $z^2 - \alpha z + \beta - 2$ with

$$\alpha = \frac{1}{\theta\tau}[v(\theta + \tau + 1) + (\theta + \tau)^2],$$

$$\beta = \frac{1}{\theta\tau}[-v - v(1 + \theta + \tau)^2 + 2\theta^2 + 2\theta\tau + 2\tau^2]$$

and

$$y = \frac{1}{(x - x^{-1})} \left(\frac{\theta\tau x - 1}{(\theta + 1)(\tau + 1)}(x + x^{-1} - 2 + v) - (v - 2)x - 2 \right).$$

Proof. We use ℓ to denote valency $v - 1 - k$ of the complement of X . Then the eigenvalues of A_2 are $v - 1 - k$, $-1 - \tau$ and $-1 - \theta$ and the equation $WW^{(-)T} = vI$ is equivalent to

$$\begin{aligned} (1 + kx + \ell y)(1 + kx^{-1} + \ell y^{-1}) &= v, \\ (1 + \theta x + (-\theta - 1)y)(1 + \theta x^{-1} + (-\theta - 1)y^{-1}) &= v, \\ (1 + \tau x + (-\tau - 1)y)(1 + \tau x^{-1} + (-\tau - 1)y^{-1}) &= v. \end{aligned}$$

Note that this set of equations is invariant under the substitutions

$$x \mapsto x^{-1}, \quad y \mapsto y^{-1}$$

and also under the substitutions

$$x \mapsto y, \quad y \mapsto x, \quad \theta \mapsto -\theta - 1, \quad \tau \mapsto -\tau - 1.$$

The missing details in the following calculations were performed in Maple.

If we set

$$X := x + \frac{1}{x}, \quad Y := y + \frac{1}{y}, \quad Z := \frac{x}{y} + \frac{y}{x}$$

then, from our three equations we get

$$\begin{aligned} k\ell Z + kX + \ell Y &= v - 1 - k^2 - \ell^2, \\ -\theta(\theta + 1)Z + \theta X - (\theta + 1)Y &= v - 1 - \theta^2 - (\theta + 1)^2, \\ -\tau(\tau + 1)Z + \tau X - (\tau + 1)Y &= v - 1 - \tau^2 - (\tau + 1)^2. \end{aligned} \tag{6}$$

These three equations are linearly dependent: if θ has multiplicity m and τ has multiplicity n as an eigenvalue of A_1 , then the first equation plus m

times the second plus n times the third is zero. In fact, our three equations are equivalent to the following pair.

$$Y - 2 + v = \frac{\theta\tau}{(\theta + 1)(\tau + 1)}(X - 2 + v), \quad (8)$$

$$Z - 2 = \frac{1}{(\theta + 1)(\tau + 1)}(X - 2 + v). \quad (9)$$

Given the definitions of Y and Z , we can view this as a pair of linear equations in y and y^{-1} , whence we find that

$$y(x - x^{-1}) = \frac{\theta\tau x - 1}{(\theta + 1)(\tau + 1)}(x + x^{-1} - 2 + v) - (v - 2)x - 2.$$

Assume $x^2 \neq 1$. If we define

$$p(x) := \tau\theta x^3 + (1 - v + 2\theta + 2\tau - \theta v - \tau v)x^2 - (2\theta + \tau\theta + 2\tau + v)x - 1,$$

then (8) and (9) hold if and only if

$$y = \frac{p(x)}{(\theta + 1)(\tau + 1)(x^2 - 1)}, \quad y^{-1} = \frac{-x^2 p(x^{-1})}{(\theta + 1)(\tau + 1)(x^2 - 1)}.$$

Then the previous expressions for y and y^{-1} hold if and only if

$$-x^2 p(x)p(x^{-1}) = [(\theta + 1)(\tau + 1)(x^2 - 1)]^2.$$

We deduce that x must be a root of the polynomial

$$(x^2 + (v - 2)x + 1)(x^4 - \alpha x^3 + \beta x^2 - \alpha x + 1) \quad (10)$$

where

$$\begin{aligned} \alpha &= \frac{1}{\theta\tau}[v(\theta + \tau + 1) + (\theta + \tau)^2], \\ \beta &= \frac{1}{\theta\tau}[-v - v(1 + \theta + \tau)^2 + 2\theta^2 + 2\theta\tau + 2\tau^2]. \end{aligned}$$

If x is a root of the quadratic factor in (10), then $X - 2 + v = 0$ and so Equations (8) and (9) imply that $Y = 2 - v$ and $Z = 2$. Since

$$Z - 2 = \frac{(x - y)^2}{xy},$$

it follows that

$$y = x = \frac{1}{2}(2 - v \pm \sqrt{v^2 - 4v}).$$

This is the Potts model solution.

We turn to the quartic factor in (10), which is equal to

$$x^2 \left((x + x^{-1})^2 - \alpha (x + x^{-1}) + \beta - 2 \right).$$

From this we see that X must be a zero of the quadratic

$$z^2 - \alpha z + \beta - 2 \tag{11}$$

and thus (d) holds.

To complete the proof we consider the cases where $x^2 = 1$. If $x = 1$ then Theorem 6.1 yields that A_2 is the incidence matrix of a symmetric design. So we assume $x = -1$.

Equations (8) and (9) imply that

$$Y - 2 + v = \theta\tau(Z - 2).$$

Since $Z = -Y$ if $x = -1$, we find that

$$(1 + \theta\tau)Y = 2 - 2\theta\tau - v$$

whence

$$y = \frac{1}{2}(\lambda \pm \sqrt{\lambda^2 - 4}),$$

where

$$\lambda = \frac{2 - 2\theta\tau - v}{1 + \theta\tau}.$$

(The denominator cannot be zero because $\tau \leq -2$ and $\theta \geq 1$ for any primitive strongly regular graph.)

If $x = -1$ then $Z = -Y$ and $X = -2$; if we add equations (8) and (9) we get

$$v - 4 = \frac{(\theta\tau + 1)(v - 4)}{(\theta + 1)(\tau + 1)}.$$

whence we find that $v - 4$ or $\theta + \tau = 0$. Since, for any strongly regular graph,

$$A^2 - (\theta + \tau)A + \theta\tau I = (k + \theta\tau)J,$$

we see that if $\theta + \tau = 0$, then $A^2 = -\theta\tau I + (k + \theta\tau)J$. Therefore A is the incidence matrix of a symmetric design (with zero diagonal and symmetric incidence matrix). \square

Jaeger [3] showed that if W is a spin model then X is formally self-dual. If X is formally self-dual then $v = (\theta - \tau)^2$ and the quadratic (11) becomes

$$\left(z - \frac{\tau^2 - \theta^2 + 2\tau}{\theta}\right) \left(z - \frac{\theta^2 - \tau^2 + 2\theta}{\tau}\right).$$

In addition to the Potts model solutions, Equations (8) and (9) give

$$\begin{aligned} x &= \frac{1}{2\tau} \left(\theta^2 - \tau^2 + 2\theta \pm \sqrt{(\theta - \tau)(\theta - \tau + 2)(\theta + \tau)(\theta + \tau + 2)} \right) \quad \text{and} \\ y &= \frac{1}{2(\theta + 1)} \left(\theta^2 - \tau^2 + 2(\theta + 1) \pm \sqrt{(\theta - \tau)(\theta - \tau + 2)(\theta + \tau)(\theta + \tau + 2)} \right), \end{aligned}$$

or

$$\begin{aligned} x &= \frac{1}{2\theta} \left(\tau^2 - \theta^2 + 2\tau \pm \sqrt{(\theta - \tau)(\theta - \tau - 2)(\theta + \tau)(\theta + \tau + 2)} \right) \quad \text{and} \\ y &= \frac{1}{2(\tau + 1)} \left(\tau^2 - \theta^2 + 2(\tau + 1) \pm \sqrt{(\theta - \tau)(\theta - \tau - 2)(\theta + \tau)(\theta + \tau + 2)} \right). \end{aligned}$$

Hence there are at most six type-II matrices, up to equivalence, in the Bose-Mesner algebra of a formally self-dual strongly regular graph.

We now determine what happens to the imprimitive strongly regular graphs, which will arise in the next section.

8.2 Theorem. *Let A_1 be the adjacency matrix of mK_{k+1} and A_2 the adjacency matrix of its complement. Suppose*

$$W := I + xA_1 + yA_2.$$

Then W is a type-II matrix if and only if one of the following holds

(a) *W is equivalent to the Potts model,*

(b)

$$x = \frac{(kv - 2k - 1)y^2 - (v - 2k - 2)y - 1}{k(1 - y^2)}$$

and

$$y + y^{-1} = \frac{2(k + 1)^2 - v(k^2 + 1)}{(k + 1)^2 - kv}$$

where $v = m(k + 1)$.

Proof. The eigenvalues of A_1 are k and -1 , so $\theta = k$ and $\tau = -1$. The equation $WW^{(-)T} = vI$ are equivalent to Equations (6) and (7):

$$\begin{aligned} -k(k+1)Z + kX - (k+1)Y &= v - 1 - k^2 - (k+1)^2 \\ X &= -v + 2. \end{aligned}$$

Solving this as a pair of linear equations in x and x^{-1} gives

$$k(1 - y^2)x = (kv - 2k - 1)y^2 - (v - 2k - 2)y - 1.$$

Assume $y^2 \neq 1$. Then Equations (6) and (7) are equivalent to

$$x = \frac{p(y)}{k(1 - y^2)}$$

and

$$x^{-1} = \frac{-y^2 p(y^{-1})}{k(1 - y^2)}$$

where

$$p(y) = (kv - 2k - 1)y^2 - (v - 2k - 2)y - 1.$$

Now these expressions for x and x^{-1} hold if and only if

$$-y^2 p(y^{-1})p(y) = k^2(1 - y^2)^2.$$

We deduce that y must be a root of the quartic

$$(y^2 + (v - 2)y + 1)(y^2 - \beta y + 1)$$

where

$$\beta = \frac{2(k+1)^2 - v(k^2 + 1)}{(k+1)^2 - kv}.$$

If y is a root of $y^2 + (v - 2)y + 1$ then we deduce from Equation (8) that $x = y$ and W is the Potts model.

If $y = 1$ then $Y = 2$, $Z = X$ and Equation (6) becomes $X = \frac{-v}{k^2} + 2$. Equations (6) and (7) imply $k = 1$. In this case, A_1 is a permutation matrix and $W = J + (x - 1)A_1$ is equivalent to the Potts model.

If $y = -1$ then $Y = -2$, $Z = -X$ and Equation (6) becomes

$$X = \frac{v - 2k^2 - 4k - 4}{k^2 + 2k}.$$

Equations (6) and (7) imply

$$2 - v = \frac{v - 2k^2 - 4k - 4}{k^2 + 2k},$$

which leads to $v = 4$ and $x = -1$. In this case, $-W = J - 2I$ is the Potts model. \square

9 Covers of Complete Graphs

Now we know that the Bose-Mesner algebra of an association scheme with two classes contains type-II matrices different from the Potts models. Given this, it is natural to ask what happens in schemes with more than two classes; in this section we consider the next simplest case. We will see that non-trivial type-II matrices do arise, and that the amount of effort required to establish this increases considerably.

We say a graph of diameter d is *antipodal* if whenever u , v and w are vertices and

$$\text{dist}(u, v) = \text{dist}(v, w) = d,$$

then $u = w$ or $\text{dist}(u, w) = d$. If X is antipodal, then the relation “at distance 0 or d ” is an equivalence relation. The cube and the line graph of the Petersen graph provide two examples with $d = 3$. If X is antipodal with $d = 2$, then it is the complement of a collection of complete graphs. If X is an antipodal graph with diameter d , then its ‘antipodal classes’ form the vertices of a distance-regular graph with the same valency and diameter $\lfloor \frac{d}{2} \rfloor$.

Here we are interested in distance-regular antipodal graphs with diameter three. To each such graph there is a set of four parameters (n, r, a_1, c_2) . The integer n is the number of antipodal classes, and r is the number of vertices in each class. If (u, v) is a pair of vertices from X and $\text{dist}(u, v) = 1$ then u and v have exactly a_1 common neighbours; if $\text{dist}(u, v) = 2$ they have exactly c_2 common neighbours. The value of a_1 is determined by n , r and c_2 , so it is conventional to provide only the triple (n, r, c_2) .

9.1 Theorem. *Suppose X is an antipodal distance regular graph of diameter three with parameters (n, r, c_2) and let A_i be the i -th distance matrix of X , for $i = 1, 2, 3$. Then the matrix*

$$W = I + xA_1 + yA_2 + zA_3$$

is type-II if and only if

- (a) $x = y$ and W is a type-II matrix in the Bose-Mesner algebra of rK_n .
- (b) $y = -x^{-1}$ and x is a solution of a quadratic equation.
- (c) $y \neq -x^{-1}$ and the possible values of (x, y) are the points of intersection of two quartics in x and y .

Proof. We use θ and τ to denote eigenvalues of X not equal to -1 or $n-1$. Now W is a type-II matrix if and only if the following system of equations are satisfied:

$$(1 - x - (r-1)y + (r-1)z) \left(1 - \frac{1}{x} - \frac{(r-1)}{y} + \frac{(r-1)}{z} \right) = nr, \quad (12)$$

$$(1 + \theta x - \theta y - z)(1 + \theta x^{-1} - \theta y^{-1} - z^{-1}) = nr, \quad (13)$$

$$(1 + \tau x - \tau y - z)(1 + \tau x^{-1} - \tau y^{-1} - z^{-1}) = nr. \quad (14)$$

Subtracting (14) from (13) gives

$$(x-y)z^{-1} + (x^{-1}-y^{-1})z = (x-y) + (x^{-1}-y^{-1}) + (\theta+\tau)(x-y)(x^{-1}-y^{-1}). \quad (15)$$

Adding θ times this to (13) yields

$$z^{-1} + z = -\theta\tau(x-y)(x^{-1}-y^{-1}) + 2 - nr. \quad (16)$$

Solving (15) and (16) as two linear equations in z and z^{-1} , we get

$$(x-y) \left((1+xy)z - \theta\tau(x-y)^2 - (\theta+\tau)(x-y) + (nr-1)xy - 1 \right) = 0. \quad (17)$$

There are three cases. First if $x = y$ we are lead to type-II matrices contained in the Bose-Mesner algebra of rK_n (including the Potts models). Second, if $xy = -1$ then (17) yields a quadratic in $X := x + x^{-1}$:

$$-\theta\tau X^2 - (\theta+\tau)X - nr = 0 \quad (18)$$

and (16) gives

$$\begin{aligned} z^{-1} + z &= -\theta\tau X^2 - nr + 2 \\ &= (\theta+\tau)X + 2. \end{aligned} \quad (19)$$

Solving (12) and (19) as two linear equations in z and z^{-1} gives

$$z = \frac{p(x)}{rx(x+1)(x-1)(r-1)}$$

and

$$z^{-1} = \frac{-x^4 p(x^{-1})}{rx(x+1)(x-1)(r-1)}$$

where

$$\begin{aligned} p(x) = & (r-1)(\theta + \tau + 1)x^4 + (\theta + \tau + r - r\theta - r\tau)x^3 + \\ & (3r\theta - r^2\tau - r^2\theta + 3r - r\theta\tau + 3r\tau - 2 - 2\theta - 2\tau - 2r^2)x^2 + \\ & (-r\theta - r\tau + 3r + \theta + \tau - 2r^2)x - (r-1)(r\theta + r\tau - 1 - \tau - \theta). \end{aligned}$$

Now these expressions for z and z^{-1} hold if and only if

$$-x^4 p(x^{-1})p(x) = [rx(x+1)(x-1)(r-1)]^2$$

which gives a quartic in X . Applying (18) to this quartic, we can express $X = x + x^{-1}$ in r , θ , and τ . Hence x is a solution of a quadratic equation.

Finally if $x \neq y$ or $-y^{-1}$, Equations (15) and (16) are equivalent to

$$z = \frac{1}{(1+xy)} (\theta\tau(x-y)^2 + (\theta + \tau)(x-y) - (nr-1)xy + 1),$$

and

$$z^{-1} = \frac{1}{xy(1+xy)} (\theta\tau(x-y)^2 - (\theta + \tau)(x-y)xy - (nr-1)xy + x^2y^2).$$

Now substituting these two expressions into (12) gives a quartic in variables x and y while $zz^{-1} = 1$ gives another one. \square

Note that rK_n is a strongly regular graph, so the possible type-II matrices are determined by the results of the previous section.

Calculations performed in Maple showed that the resultant with respect to x of the two quartics in case (c) is a non-zero polynomial in y of degree at most 30. By the elimination property of resultants [2], the resultant vanishes at any common solution of the two quartics. Hence these two quartics vanish at no more than thirty values for y . Similarly, the resultant with respect to y of these two quartics is a non-zero polynomial in x of degree at most 30

and they vanish at no more than thirty values for x . Consequently there are finitely many type-II matrices, up to scalar multiplication, in the Bose-Mesner algebra of an antipodal distance regular graph of diameter three.

As a final remark, it could be true that each Bose-Mesner algebra is equal to the set of all polynomials in some type-II matrix. The results of the last two sections imply this is true for schemes with at most two classes, and for antipodal schemes with three classes. (Since we do not have strong evidence either way, we will not make any conjecture.)

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